Five common fallacies concerning Length Contraction and Time Dilation

Fallacy No 1: *Length Contraction and Time Dilation are complimentary; one effect causes an increase while the other effect causes a decrease; they therefore cancel each other out which is why all observers measure the same value for the speed of light.*

This is a classic case of assuming that if you get the right answer, the reasoning must be correct. You could just as easily argue that, since speed = distance over time, if distance decreases and time increases, then the speed of light must get smaller!

What, then, is the correct argument?

The coordinates of an event in a 'moving' frame B (t', x', y', z') are related to the coordinates of the same event in the 'stationary' frame A (t, x, y, z) whose axes are aligned and which share the same origin by the following equations known as the Lorentz transformation:

$$\begin{aligned} t' &= \gamma(t - vx/c^2) \\ x' &= \gamma(x - vt) \\ y' &= y \\ z' &= z \end{aligned} \qquad (1)$$

where v is the relative velocity between the two frames and

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} \qquad (2)$$

If a light beam is sent off from the origin (0, 0, 0, 0) along the X axis it will reach the point (t, ct, 0, 0) in the stationary frame A. The coordinates of this event in frame B are:

$$\begin{aligned} t' &= \gamma(t - vct/c^2) = \gamma t(1 - v/c) \\ x' &= \gamma(ct - vt) = \gamma t(c - v) \\ y' &= y \\ z' &= z \end{aligned}$$

The ratio x' / t' is the speed which the light ray moves in frame B so:

$$\frac{x'}{t'} = \frac{\gamma t(c - v)}{\gamma t(1 - v/c)} = c$$

It is true that the gamma factor cancels out but not because it causes an increase in the distance and a decrease in the time but precisely because it causes the same increase in both coordinates!

Fallacy No 2: *Time Dilation occurs in a moving frame because the light rays have further to go along their zig-zag route which causes time to slow down by a factor γ. Obviously, in order to make light travel in both frames at the same speed, lengths have to be decreased by the same factor.*

This is essentially the same argument but looked at from a different starting point. Any intelligent reader would have to say – hang on, if the speed is the same, shouldn't the lengths be increased, not decreased – thus contributing further to the confusion!

Now the argument that shows that if time is dilated, lengths are contracted is by no means obvious and, in fact, when we say that time is dilated (i.e. expanded) and lengths are contracted, we are really talking about two different things. So lets look carefully at some events which occur in the 'moving' frame B as seen in the 'stationary' frame A. First we must turn the equations in (1) round so that x and t are expressed in terms of x' and t' etc. I will not go through the algebra but if you choose to do it yourself you will discover a wonderful thing – this is the result:

$$\begin{aligned} t &= \gamma(t' + vx'/c^2) \\ x &= \gamma(x' + vt') \\ y &= y \\ z &= z \end{aligned} \tag{3}$$

The equations look exactly the same – except that the minus sign has been replaced by a plus sign.

This is a very important result and it shows that the situation is entirely symmetrical with respect to who is actually moving, the only difference being that if frame B is moving with speed $+v$ in frame A, frame A is moving with speed $-v$ in frame B. (It is worth noting that this transformation only works if $\gamma = \dfrac{1}{\sqrt{1 - v^2/c^2}}$).

Now let us look at Time Dilation. Suppose that on board the spaceship in frame B the pilot sets off a firecracker at (0, 0, 0, 0) and then a second firecracker exactly 1 second later at (1, 0, 0, 0). Using equations (3) we find that the temporal coordinate of the second event in frame A is $\gamma t = \gamma$. i.e. in frame A the event occurs at a later time (because γ is always greater than 1). It is true that the x coordinate is also different but this does not alter the fact that, measured in frame A, 1 second is expanded to γ seconds in the 'stationary' frame.

Now what about distances?

Suppose that two firecrackers are set off at exactly the same time in B's frame, one at (0, 0, 0, 0) and the other at 0, 1, 0, 0) i.e. one is 1 metre in front of the other in the direction of motion at the ends of a metre ruler. These coordinates translate into (0, 0, 0, 0) and ($\gamma v/c^2$, γ, 0, 0) in A's frame. Just as time was expanded in the previous example, the front firecracker goes of γ metres down the axis. In other words, the distance coordinate has been *increased* by the gamma factor – not decreased.

What has gone wrong? Well, the first thing to say is that the X coordinate is not the length of the metre ruler. In order to calculate the latter we must calculate where the front of the ruler was when the rear firecracker went off (in A's frame). In other words we must subtract the distance which the metre ruler has moved in the time $\gamma v/c^2$. This distance is $\gamma v^2/c^2$ and the measured length of the ruler will be $\gamma - \gamma v^2/c^2 = \gamma(1 - v^2/c^2)$ hence:

$$\text{length of ruler} = \frac{1}{\sqrt{1 - v^2/c^2}} \times (1 - v^2/c^2) = \sqrt{1 - v^2/c^2} = \frac{1}{\gamma}$$

You can see how easy it is to make a mistake with all those reciprocals and how easy it is to fall into the trap of saying that the result is obvious when it is not obvious at all!

Fallacy No 3: *In the Michelson Morley experiment two light beam are set to race at right angles to one another. The beam which travels across the direction of motion takes longer then it should by the gamma factor because it has to 'head upstream'. On the other hand, the beam which travels parallel to the direction of motion is helped as much as it is hindered by the motion of the apparatus through space so one of the beams has to be made shorter by the factor gamma to compensate in order that the beams arrive back together.*

If this argument was correct it would be the arm of the apparatus which was at right angles to the motion which was made shorter, not the one which is parallel. The mistake in the argument lies in the claim that the beam which travels parallel to the motion is 'helped as much as it is hindered'. In fact, the time taken to travel 'upstream' a distance D and back again is equal to

$$t_{\text{parallel}} = \frac{D}{c + v} + \frac{D}{c - v} = \frac{2cD}{c^2 - v^2}$$

The time taken for a beam to travel across the motion and back again is equal to

$$t_{\text{across}} = \frac{2D}{\sqrt{c^2 - v^2}}$$

It turns out that t_{parallel} is always longer then t_{across} and therefore the parallel leg must be made shorter by the factor

$$\frac{t_{\text{parallel}}}{t_{\text{across}}} = \frac{2cD}{c^2 - v^2} \times \frac{\sqrt{c^2 - v^2}}{2D} = \frac{1}{\sqrt{1 - v^2/c^2}} = \gamma$$

Once again, the algebra is just a little more complicated than it would appear at first sight.

Much of the confusion in the minds of beginners to Special Relativity can be traced back to the fact that the gamma factor is the reciprocal of a quantity which is less than 1. When asked to divide and expression by gamma, it is all too easy to forget which way up the expression is and end up multiplying instead of dividing.

Fallacy No 4: *When a spaceship passes you at near light speed it looks shorter according to Special Relativity but this is only an illusion caused by your particular perspective on the situation.*

There are two errors in this statement. The first is that Special Relativity does not tell us what the spaceship *looks* like but what it *is.* In fact, even if we ignore the effects of Special Relativity, the fact that light travels at a finite speed will make a spaceship appear *longer* when it is approaching us and *shorter* when it is receding.

Secondly, Special Relativity tells us that the actual length of the spaceship *really is contracted*. In spite of the fact that different observers will witness different amounts of contraction (including the occupants of the spaceship who will not witness any contraction at all) in each observers' frame, the spaceship is physically reduced in length and we must seek a physical explanation for the effect. (I discuss this issue in greater detail in my article *The Reality of Length Contraction and Time Dilation*.

Fallacy No 5: *The Twin's Paradox is explained because it is the travelling twin who experiences decelerations and accelerations when he turns round and this causes his clocks to slow down during the turn round. It is therefore necessary to use General Relativity to explain why he is younger than his brother when he returns.*

The effect is explained perfectly using Special Relativity alone. Neither acceleration nor gravity causes clocks to slow down.

Carr Bank, December 2021